On Beltrami fields with nonconstant proportionality factor

2003 J. Phys. A: Math. Gen. 36 1515

(http://iopscience.iop.org/0305-4470/36/5/323)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 1515–1522 PII: S0305-4470(03)53762-3

On Beltrami fields with nonconstant proportionality factor

Vladislav V Kravchenko

Depto de Telecomunicaciones, SEPI, Escuela Superior de Ingeniería Mecánica y Eléctrica, Instituto Politécnico Nacional, CP 07738 México DF, Mexico

Received 19 September 2002
Published 22 January 2003
Online at stacks.iop.org/JPhysA/36/1515

Abstract
The equation $\mathrm{rot}\,\vec{f}(x)+\alpha(x)\vec{f}(x)=0$ is considered, where α is a nonvanishing complex valued function. Its quaternionic reformulation is obtained which is used for constructing integral representations for solutions in the case when α is a function of one variable. We show that in this case the solution of the considered equation reduces to the solution of three different Schrödinger equations with potentials depending on one variable.

PACS numbers: 02.30.Em, 02.30.Jr

1. Introduction

Solutions of the equation

$$\mathrm{rot}\,\vec{f}(x)+\alpha(x)\vec{f}(x)=0 \tag{1}$$

where α is a complex valued function of space coordinates $x=(x_1,x_2,x_3)$ are known as Beltrami fields and appear in different branches of modern physics (see, e.g., [1, 4–6, 12, 15, 16]). When α is constant much information about (1) is available, including integral representations for the solutions (see, e.g., [11, 12]), solutions of some boundary value problems (see, e.g., [1, 11, 13]) and spectral problems [14]. The situation is clearly much more complicated when α is a function.

In this work we apply a quaternionic approach for the analysis of (1) and study in detail the case when α is a complex valued function of one coordinate: $\alpha=\alpha(x_1)$. The main result consists in the reduction of the problem of obtaining solutions of (1) to the solution of some Schrödinger equations, the theory of which is better developed. Based on the solutions of three different Schrödinger equations we obtain an integral representation for solutions of (1).

2. Preliminaries

We will consider the algebra of complex quaternions which have the form $q=\sum_{k=0}^{3}q_k i_k$ where $\{q_k\}\subset\mathbb{C}$, i_0 is the unit and $\{i_k\,|\,k=1,2,3\}$ are the quaternionic imaginary units, that is the standard basis elements possessing the following properties:

$$i_0^2 = i_0 = -i_k^2 \qquad i_0 i_k = i_k i_0 = i_k \qquad k = 1, 2, 3$$
$$i_1 i_2 = -i_2 i_1 = i_3 \qquad i_2 i_3 = -i_3 i_2 = i_1 \qquad i_3 i_1 = -i_1 i_3 = i_2.$$

We denote the imaginary unit in $\mathbb{C}$ by i as usual. By definition i commutes with i_k, $k = \overline{0, 3}$.

The vectorial representation of a complex quaternion will be used. Namely, each complex quaternion q is a sum of a scalar q_0 and of a vector $\vec{q}$:

$$q = \mathrm{Sc}(q) + \mathrm{Vec}(q) = q_0 + \vec{q}$$

where $\vec{q} = \sum_{k=1}^{3} q_k i_k$. The purely vectorial complex quaternions ($\mathrm{Sc}(q) = 0$) are identified with vectors from $\mathbb{C}^3$.

By M^p we denote the operator of multiplication by the complex quaternion p from the right-hand side: $M^p q = q \cdot p$.

We will intensively use the fact that the algebra of complex quaternions contains a subset of zero divisors which are characterized by the equality $q_0^2 = \vec{q}^2$, where $\vec{q}^2 = -\langle \vec{q}, \vec{q} \rangle$, or equivalently $q^2 = 2q_0 q$. We see that if q is a zero divisor and $q_0 = 1/2$ then q is an idempotent. More information on the structure of the algebra of complex quaternions can be found for example in [11].

Let f be a complex quaternion valued differentiable function of $x = (x_1, x_2, x_3)$. Denote

$$Df = \sum_{k=1}^{3} i_k \frac{\partial}{\partial x_k} f.$$

This expression can be rewritten in vector form as follows:

$$Df = -\mathrm{div}\vec{f} + \mathrm{grad}\, f_0 + \mathrm{rot}\vec{f}.$$

That is, $\mathrm{Sc}(Df) = -\mathrm{div}\vec{f}$ and $\mathrm{Vec}(Df) = \mathrm{grad}\, f_0 + \mathrm{rot}\vec{f}$.

Let us note two properties of the operator D. It factorizes the Laplace operator: $D^2 = -\Delta$. For a scalar function ϕ we have

$$\left(D + \frac{\mathrm{grad}\,\phi}{\phi}\right) f = \phi^{-1} D(\phi f). \tag{2}$$

3. Quaternionic reformulation of (1)

If α is constant then (1) is equivalent to the equation

$$(D + \alpha)\vec{f} = 0. \tag{3}$$

The same is not true in the case when α is a function because from (1) we obtain that

$$\alpha\, \mathrm{div}\vec{f} + \langle \mathrm{grad}\,\alpha, \vec{f} \rangle = 0 \tag{4}$$

but the scalar part of (3) gives us $\mathrm{div}\vec{f} = 0$.

Nevertheless (1) can be rewritten in a quite convenient quaternionic form as is shown in the following statement. We assume that α is a nonvanishing function and fix a branch of $\sqrt{\alpha}$, for example the positive on the positive real semiaxis.

Proposition 1. *A $\mathbb{C}^3$-valued function $\vec{f}$ is a solution of (1) if and only if the purely vectorial complex quaternionic function $\vec{g} = \sqrt{\alpha}\vec{f}$ is a solution of the equation*

$$(D + M^{\alpha+\vec{\gamma}})\vec{g} = 0 \tag{5}$$

where $\vec{\gamma} = \frac{\mathrm{grad}\sqrt{\alpha}}{\sqrt{\alpha}}$.

Proof. Consider the equation

$$(D + \alpha + \vec{\gamma} + M^{\vec{\gamma}})\vec{f} = 0. \tag{6}$$

It is easy to see that its vector part coincides with (1) and the scalar part gives us

$$\operatorname{div}\vec{f} + 2\left\langle \frac{\operatorname{grad}\sqrt{\alpha}}{\sqrt{\alpha}}, \vec{f} \right\rangle = 0$$

which is equivalent to (4). Thus (6) is equivalent to (1).

Now, using (2) we rewrite (6) in the following form:

$$\frac{1}{\sqrt{\alpha}} D(\sqrt{\alpha}\vec{f}) + \alpha\vec{f} + \vec{f}\vec{\gamma} = 0.$$

Multiplying this equation by $\sqrt{\alpha}$ and introducing the notation $\vec{g} = \sqrt{\alpha}\vec{f}$ we obtain the required fact. $\square$

Remark 2. The operator $D + M^{\alpha+\vec{\gamma}}$ is closely related to the classical Dirac operator. For example, let $\vec{\gamma}(x) = -\mathrm{i}(\omega + \phi_{\mathrm{el}}(x))i_1 - (m - \mathrm{i}\phi_{sc}(x))i_2$. Then the equation $(D + M^{\alpha+\vec{\gamma}})g = 0$ is equivalent (see [7–9]) to the equation

$$\left(\mathrm{i}\omega\gamma_0 + \sum_{k=1}^{3}\gamma_k\frac{\partial}{\partial x_k} + \mathrm{i}m + \mathrm{i}\gamma_0\gamma_5\alpha(x) + \phi_{\mathrm{sc}}(x) + \mathrm{i}\gamma_0\phi_{\mathrm{el}}(x)\right)\Phi(x) = 0$$

where γ_k are standard γ-matrices, α is called the pseudoscalar potential, ϕ_{sc} is the scalar potential and ϕ_{el} is the electric potential.

Remark 3. The Maxwell system

$$\operatorname{div}(\varepsilon(x)\vec{E}(x)) = 0 \qquad \text{and} \qquad \operatorname{rot}\vec{E}(x) = 0$$

is equivalent to the equation

$$(D + M^{\vec{\epsilon}(x)})\,\mathbf{E}(x) = 0$$

where $\mathbf{E} = \sqrt{\varepsilon}\vec{E}$ and $\vec{\epsilon} = \frac{\operatorname{grad}\sqrt{\varepsilon}}{\sqrt{\varepsilon}}$ (see [10]).

4. α is a function of one variable

Assume that $\alpha = \alpha(x_1)$. According to proposition 1 equation (1) is equivalent to the quaternionic equation

$$\left(D + M^{(\alpha + \frac{\alpha'}{2\alpha}i_1)}\right)\vec{g} = 0.$$

Thus we are interested in solutions of the equation

$$\left(D + M^{(\alpha+\gamma i_1)}\right)g = 0 \tag{7}$$

where α and γ are complex valued functions of x_1 and in general we will consider not only purely vectorial solutions of (7) but complete complex quaternions g.

Denote $P^{\pm} = \frac{1}{2}M^{(1\pm\mathrm{i}i_1)}$.

Proposition 4.

1. *The following equality is true*

$$D + M^{\alpha+\gamma i_1} = P^{+}(D + M^{(\gamma+\mathrm{i}\alpha)i_1}) + P^{-}(D + M^{(\gamma-\mathrm{i}\alpha)i_1}). \tag{8}$$

2. *Any solution of (7) has the form $g = P^{+}v + P^{-}w$, where v is a solution of the equation*

$$(D + M^{(\gamma+\mathrm{i}\alpha)i_1})v = 0$$

and w is a solution of the equation

$$(D + M^{(\gamma-\mathrm{i}\alpha)i_1})w = 0.$$

Proof. In order to verify (8) it is sufficient to note that

$$\alpha = P^+\alpha + P^-\alpha = \tfrac{1}{2}(\alpha(1 + \mathrm{i}i_1) + \alpha(1 - \mathrm{i}i_1)) = \tfrac{1}{2}(\mathrm{i}\alpha i_1(1 + \mathrm{i}i_1) - \mathrm{i}\alpha i_1(1 - \mathrm{i}i_1)).$$

The second statement of the proposition follows from the fact that P^+ and P^- commute with the operators $D + M^{(\gamma+\mathrm{i}\alpha)i_1}$ and $D + M^{(\gamma-\mathrm{i}\alpha)i_1}$. □

Thus the problem reduces to the study of the equation

$$D_{\vec{\beta}}u = 0$$

where $D_{\vec{\beta}} = D + M^{\beta(x_1)i_1}$.

Using the factorization of the Schrödinger operator proposed in [2, 3] we obtain the following.

Proposition 5. *Let $\mu = \beta' + \beta^2$ and $\nu = -\beta' + \beta^2$. Then for a scalar function φ we have*

$$D_{\vec{\beta}}D_{-\vec{\beta}}\varphi = (D + M^{\beta(x_1)i_1})(D - M^{\beta(x_1)i_1})\varphi = (-\Delta + \mu)\varphi \tag{9}$$

and

$$D_{-\vec{\beta}}D_{\vec{\beta}}\varphi = (-\Delta + \nu)\varphi. \tag{10}$$

Proof.

$$\begin{aligned}(D + M^{\beta i_1})(D - M^{\beta i_1})\varphi &= -\Delta\varphi - (D\varphi)\beta i_1 - \varphi D(\beta i_1) + (D\varphi)\beta i_1 + \beta^2\varphi \\ &= -\Delta\varphi + (\beta' + \beta^2)\varphi = (-\Delta + \mu)\varphi.\end{aligned}$$

Equality (10) is verified in the same way. □

Corollary 6. *Let φ be a fundamental solution of the operator $-\Delta + \mu$ and ψ be a fundamental solution of the operator $-\Delta + \nu$. Then $\mathcal{K}_{\vec{\beta}} = D_{-\vec{\beta}}\varphi$ is a fundamental solution of $D_{\vec{\beta}}$: $D_{\vec{\beta}}\mathcal{K}_{\vec{\beta}} = \delta$ and $\mathcal{K}_{-\vec{\beta}} = D_{\vec{\beta}}\psi$ is a fundamental solution of $D_{-\vec{\beta}}$: $D_{-\vec{\beta}}\mathcal{K}_{-\vec{\beta}} = \delta$.*

Usually the fundamental solution of a differential operator can be used for constructing the corresponding right inverse operator. For example, if φ is a fundamental solution of the operator $-\Delta + \mu$ then the convolution $\int_\Omega \varphi(x - y)f(y)\,\mathrm{d}y$ defines a right inverse operator corresponding to $-\Delta + \mu$ at least in a bounded domain Ω and in an appropriate functional space. With fundamental solutions of the operators $D_{\vec{\beta}}$ and $D_{-\vec{\beta}}$ the situation is more complicated. The operator D is applied from the left and the multiplication by βi_1 is from the right. Hence a simple convolution with a fundamental solution does not give us a right inverse operator. The solution consists of one additional step. Having fundamental solutions for $D_{\vec{\beta}}$ and $D_{-\vec{\beta}}$ and using the operators P^+ and P^- we can also construct fundamental solutions for the operators $D_{-\mathrm{i}\beta} = D - \mathrm{i}\beta(x_1)$ and $D_{\mathrm{i}\beta} = D + \mathrm{i}\beta(x_1)$. Here the point is that the multiplicative terms are scalars and consequently the left-sided convolutions with fundamental solutions of these operators will give us corresponding right inverse operators. Then using P^+ and P^- once more we transform them into right inverse operators for $D_{\vec{\beta}}$ and $D_{-\vec{\beta}}$.

By analogy with proposition 1 we note that

$$D - \mathrm{i}\beta = P^+(D + M^{\beta i_1}) + P^-(D - M^{\beta i_1}) \tag{11}$$

and

$$D + \mathrm{i}\beta = P^+(D - M^{\beta i_1}) + P^-(D + M^{\beta i_1}). \tag{12}$$

Let $\mathcal{K}_{\vec{\beta}}$ be a fundamental solution for $D + M^{\beta i_1}$ and $\mathcal{K}_{-\vec{\beta}}$ be a fundamental solution for $D - M^{\beta i_1}$. Then from (11) we obtain that

$$\mathcal{K}_{-\mathrm{i}\beta} = P^+\mathcal{K}_{\vec{\beta}} + P^-\mathcal{K}_{-\vec{\beta}} \tag{13}$$

is a fundamental solution of the operator $D_{-\mathrm{i}\beta}$ and from (12) we obtain that

$$\mathcal{K}_{\mathrm{i}\beta} = P^{+}\mathcal{K}_{-\vec{\beta}} + P^{-}\mathcal{K}_{\vec{\beta}}$$

is a fundamental solution of the operator $D_{\mathrm{i}\beta}$.

Let Ω be a bounded domain in $\mathbb{R}^3$ with a closed Liapunov boundary Γ. Let $T_{\pm\mathrm{i}\beta}$ and $K_{\pm\mathrm{i}\beta}$ denote the operators acting on complex quaternion valued functions by the following rules:

$$T_{\pm\mathrm{i}\beta}g(x) = \int_{\Omega} \mathcal{K}_{\pm\mathrm{i}\beta}(x-y)g(y)\,\mathrm{d}y \qquad x \in \mathbb{R}^3$$

and

$$K_{\pm\mathrm{i}\beta}g(x) = -\int_{\Gamma} \mathcal{K}_{\pm\mathrm{i}\beta}(x-y)\vec{n}(y)g(y)\,\mathrm{d}\Gamma_y \qquad x \in \mathbb{R}^3\backslash\Gamma$$

where $\vec{n}$ is the outward unit normal $\vec{n} = \sum_{k=1}^{3} n_k i_k$.

Theorem 7 (Borel–Pompeiu formula)**.** *Let g be a complex quaternion valued function with components belonging to $C^1(\Omega) \cap C(\overline{\Omega})$. Then*

$$K_{\pm\mathrm{i}\beta}g(x) + T_{\pm\mathrm{i}\beta}D_{\pm\mathrm{i}\beta}g(x) = g(x) \qquad x \in \Omega.$$

Proof. This proof is completely analogous to that from [11, p 68] (where it was given for a constant β) and is based on a quaternionic version of the Stokes formula. □

From this theorem two corollaries follow immediately (compare with [11, pp 70, 71]).

Theorem 8 (Cauchy's integral formula)**.** *Under the conditions of theorem 7, let g be a solution of the equation $D_{\mathrm{i}\beta}g = 0$ or $D_{-\mathrm{i}\beta}g = 0$ in Ω. Then $g(x) = K_{\mathrm{i}\beta}g(x)$ or $g(x) = K_{-\mathrm{i}\beta}g(x)$, $x \in \Omega$ respectively.*

Theorem 9 (Right inverse operator)**.** *Under the conditions of theorem 7 the following equality holds:*

$$D_{\pm\mathrm{i}\beta}T_{\pm\mathrm{i}\beta}g(x) = g(x) \qquad x \in \Omega.$$

Now let us turn back to the operator $D_{\vec{\beta}}$. From (11) and (12) we have

$$D_{\vec{\beta}} = P^{+}D_{-\mathrm{i}\beta} + P^{-}D_{\mathrm{i}\beta}.$$

Introducing the notation

$$T_{\vec{\beta}} = P^{+}T_{-\mathrm{i}\beta} + P^{-}T_{\mathrm{i}\beta} \tag{14}$$

and

$$K_{\vec{\beta}} = P^{+}K_{-\mathrm{i}\beta} + P^{-}K_{\mathrm{i}\beta} \tag{15}$$

we obtain similar facts as those formulated in theorems 7–9.

Theorem 10. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a closed Liapunov boundary Γ, $g \in C^1(\Omega) \cap C(\overline{\Omega})$. Then*

$$K_{\vec{\beta}}g(x) + T_{\vec{\beta}}D_{\vec{\beta}}g(x) = g(x) \qquad x \in \Omega$$
$$D_{\vec{\beta}}T_{\vec{\beta}}g(x) = g(x) \qquad x \in \Omega$$

and if additionally $D_{\vec{\beta}}g = 0$ in Ω then

$$g(x) = K_{\vec{\beta}}g(x) \qquad x \in \Omega. \tag{16}$$

Let us give a more explicit form of the equality (16):

$$
\begin{aligned}
g(x) = -\frac{1}{4}\int_\Gamma \{ & ((D\varphi(x-y) - \beta(x_1-y_1)\varphi(x-y)i_1)(1+\mathrm{i}i_1) \\
& + (D\psi(x-y) + \beta(x_1-y_1)\psi(x-y)i_1)(1-\mathrm{i}i_1))\vec{n}(y)g(y)(1+\mathrm{i}i_1) \\
& + ((D\psi(x-y) + \beta(x_1-y_1)\psi(x-y)i_1)(1+\mathrm{i}i_1) \\
& + (D\varphi(x-y) - \beta(x_1-y_1)\varphi(x-y)i_1)(1-\mathrm{i}i_1))\vec{n}(y)g(y)(1-\mathrm{i}i_1)\}\,\mathrm{d}\Gamma_y
\end{aligned}
$$

where φ is a fundamental solution of the operator $-\Delta+\mu$ and ψ is a fundamental solution of the operator $-\Delta+\nu$.

For the operator $D + M^{\alpha+\gamma i_1}$ from (8) we obtain the corresponding Cauchy integral operator and the T-operator in the form

$$K_{\alpha+\gamma i_1} = P^+K_{(\gamma+\mathrm{i}\alpha)i_1} + P^-K_{(\gamma-\mathrm{i}\alpha)i_1}$$

and

$$T_{\alpha+\gamma i_1} = P^+T_{(\gamma+\mathrm{i}\alpha)i_1} + P^-T_{(\gamma-\mathrm{i}\alpha)i_1}$$

where according to (15) and (14):

$$
\begin{aligned}
K_{(\gamma+\mathrm{i}\alpha)i_1} &= P^+K_{\alpha-\mathrm{i}\gamma} + P^-K_{-(\alpha-\mathrm{i}\gamma)} \\
T_{(\gamma+\mathrm{i}\alpha)i_1} &= P^+T_{\alpha-\mathrm{i}\gamma} + P^-T_{-(\alpha-\mathrm{i}\gamma)} \\
K_{(\gamma-\mathrm{i}\alpha)i_1} &= P^+K_{-(\alpha+\mathrm{i}\gamma)} + P^-K_{\alpha+\mathrm{i}\gamma} \\
T_{(\gamma-\mathrm{i}\alpha)i_1} &= P^+T_{-(\alpha+\mathrm{i}\gamma)} + P^-T_{\alpha+\mathrm{i}\gamma}.
\end{aligned}
$$

Thus,

$$K_{\alpha+\gamma i_1} = P^+K_{\alpha-\mathrm{i}\gamma} + P^-K_{\alpha+\mathrm{i}\gamma}$$

and

$$T_{\alpha+\gamma i_1} = P^+T_{\alpha-\mathrm{i}\gamma} + P^-T_{\alpha+\mathrm{i}\gamma}.$$

For the operator $D_{\alpha+\gamma i_1} = D + M^{\alpha+\gamma i_1}$ together with these two operators we obtain again all facts from theorem 10.

Theorem 11. *Let Ω be a bounded domain in $\mathbb{R}^3$ with a closed Liapunov boundary Γ, $g \in C^1(\Omega)\cap C(\overline{\Omega})$. Then*

$$
\begin{aligned}
&K_{\alpha+\gamma i_1}g(x) + T_{\alpha+\gamma i_1}D_{\alpha+\gamma i_1}g(x) = g(x) \qquad && x\in\Omega \\
&D_{\alpha+\gamma i_1}T_{\alpha+\gamma i_1}g(x) = g(x) && x\in\Omega
\end{aligned}
$$

and if additionally $D_{\alpha+\gamma i_1}g = 0$ in Ω then

$$g(x) = K_{\alpha+\gamma i_1}g(x) \qquad x\in\Omega.$$

Finally, let us obtain similar results for solutions of (1) when $\alpha = \alpha(x_1)$. Note that according to our construction (equality (13) and proposition 5)

$$
\begin{aligned}
\mathcal{K}_{\alpha-\mathrm{i}\gamma} &= P^+\mathcal{K}_{(\gamma+\mathrm{i}\alpha)i_1} + P^-\mathcal{K}_{-(\gamma+\mathrm{i}\alpha)i_1} \\
&= P^+(D-(\gamma+\mathrm{i}\alpha)i_1)\varphi_1 + P^-(D+(\gamma+\mathrm{i}\alpha)i_1)\psi_1
\end{aligned}
$$

where φ_1 is a fundamental solution of $-\Delta+\mu_1$ with $\mu_1 = \gamma' + \mathrm{i}\alpha' + (\gamma+\mathrm{i}\alpha)^2$ and ψ_1 is a fundamental solution of $-\Delta+\nu_1$ with $\nu_1 = -\gamma' - \mathrm{i}\alpha' + (\gamma+\mathrm{i}\alpha)^2$. Analogously,

$$
\begin{aligned}
\mathcal{K}_{\alpha+\mathrm{i}\gamma} &= P^+\mathcal{K}_{-(\gamma-\mathrm{i}\alpha)i_1} + P^-\mathcal{K}_{(\gamma-\mathrm{i}\alpha)i_1} \\
&= P^+(D+(\gamma-\mathrm{i}\alpha)i_1)\varphi_2 + P^-(D-(\gamma-\mathrm{i}\alpha)i_1)\psi_2
\end{aligned}
$$

where φ_2 is a fundamental solution of $-\Delta + \mu_2$ with $\mu_2 = -\gamma' + \mathrm{i}\alpha' + (\gamma - \mathrm{i}\alpha)^2$ and ψ_2 is a fundamental solution of $-\Delta + \nu_2$ with $\nu_2 = \gamma' - \mathrm{i}\alpha' + (\gamma - \mathrm{i}\alpha)^2$.

Turning back to equation (1) we recall that $\gamma = \alpha'/(2\alpha)$. Substituting this expression into the expressions for μ_1, ν_1, μ_2 and ν_2 we find that

$$\mu_1 = \frac{1}{2}\frac{\alpha''}{\alpha} - \frac{1}{4}\frac{(\alpha')^2}{\alpha^2} + 2\mathrm{i}\alpha' - \alpha^2$$

$$\nu_1 = \mu_2 = -\frac{1}{2}\frac{\alpha''}{\alpha} + \frac{3}{4}\frac{(\alpha')^2}{\alpha^2} - \alpha^2$$

and

$$\nu_2 = \frac{1}{2}\frac{\alpha''}{\alpha} - \frac{1}{4}\frac{(\alpha')^2}{\alpha^2} - 2\mathrm{i}\alpha' - \alpha^2.$$

Thus in this particular case we have only three different Schrödinger operators. Moreover, if α is a real function then ν_2 is a complex conjugate of μ_1 and hence having solved one Schrödinger equation the solution of the other can be obtained immediately.

Assuming that the fundamental solutions of the three Schrödinger operators are given we construct the operator $K_{\alpha+\gamma i_1}$ and as a corollary of theorem 11 we obtain the following integral representation for solutions of (1).

Theorem 12. *Let $\alpha = \alpha(x_1)$ be a complex valued twice differentiable function, Ω be a bounded domain in $\mathbb{R}^3$ with a closed Liapunov boundary Γ, $\vec{f} \in C^1(\Omega) \cap C(\overline{\Omega})$ and let $\vec{f}$ satisfy (1). Then*

$$\vec{f}(x) = \frac{1}{\sqrt{\alpha}} K_{\alpha+\gamma i_1}(\sqrt{\alpha}\vec{f})(x) \qquad x \in \Omega$$

where $\gamma = \alpha'/(2\alpha)$.

This theorem allows us to reconstruct a solution of (1) in a domain Ω by its boundary values on $\Gamma = \partial\Omega$ and hence represents a first necessary step for solving boundary value problems for equation (1) in the case when α is a function.

Acknowledgment

This work was supported by CONACYT Project 32424-E, Mexico.

References

[1] Athanasiadis C, Costakis G and Stratis I G 2000 On some properties of Beltrami fields in chiral media *Rep. Math. Phys.* **45** 257–71

[2] Bernstein S 1996 Factorization of solutions of the Schrödinger equation *Proc. Symp.: Analytical and Numerical Methods in Quaternionic and Clifford Analysis (Seiffen)* ed W Sprössig and K Gürlebeck

[3] Bernstein S and Gürlebeck K 1999 On a higher dimensional Miura transform *Complex Variables* **38** 307–19

[4] Feng Qingzeng 1997 On force-free magnetic fields and Beltrami flows *Appl. Math. Mech.* **18** 997–1003 (English edn)

[5] Gonzalez-Gascon F and Peralta-Salas D 2001 Ordered behaviour in force-free magnetic fields *Phys. Lett.* A **292** 75–84

[6] Kaiser R, Neudert M and Wahl von W 2000 On the existence of force-free magnetic fields with small nonconstant α in exterior domains *Commun. Math. Phys.* **211** 111–36

[7] Kravchenko V V 1998 Exact solutions of the Dirac equation with harmonic pseudoscalar, scalar or electric potential *J. Phys. A: Math. Gen.* **31** 7561–75

[8] Kravchenko V V 2000 A new method for obtaining solutions of the Dirac equation *Z. Anal. Anwendungen* **19** 655–76

[9] Kravchenko V V 2001 On a new approach for solving Dirac equations with some potentials and Maxwell's system in inhomogeneous media *Problems and Methods in Mathematical Physics. The Siegfried Prössdorf Memorial Volume (Operator Theory: Advances and Applications vol 121)* ed J Elschner *et al* (Basel: Birkhäuser) pp 278–306

[10] Kravchenko V V 2002 Quaternionic reformulation of Maxwell's equations for inhomogeneous media and new solutions *Z. Anal. Anwendungen* **21** 21–6

[11] Kravchenko V V and Shapiro M V 1996 *Integral Representations for Spatial Models of Mathematical Physics (Pitman Res. Notes in Math. Series vol 351)* (Harlow: Addison Wesley Longman)

[12] Lakhtakia A 1994 *Beltrami Fields in Chiral Media* (Singapore: World Scientific)

[13] Saks R S 1972 On boundary value problems for the system $\mathrm{rot}\, u + \lambda u = h$ *Diff. Eqns.* **8** 126–33

[14] Saks R S 2001 On spectra of the operator rotor *Functional-Analytic and Complex Methods, their Interactions, and Applications to Partial Differential Equations* ed H Florian *et al* (Singapore: World Scientific) pp 58–63

[15] Yoshida Z 1997 Applications of Beltrami functions in plasma physics *Nonlinear Anal. Theory Appl.* **30** 3617–27

[16] Zaghloul H and Barajas O 1990 Force-free magnetic fields *Am. J. Phys.* **58** 783–8